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Local Convergence of Lagrange Interpolation Associated with Equidistant Nodes

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Under the assumption that the function f is bounded on $[-1, 1]$ and analytic at $x=0$ we prove the local convergence of Lagrange interpolating polynomials of f associated with equidistant nodes on $[-1, 1]$. The classical results concerning the convergence of such interpolants assume the stronger condition that f is analytic on $[-1, 1]$. A de Montessus de Ballore type theorem for interpolating rationals associated with equidistant nodes is also established without assuming the global analyticity of f on $[-1, 1]$.

1. INTRODUCTION

Let f be a function defined on $[-1, 1]$ and denote by $L_n(f; \cdot)$ the Lagrange interpolating polynomial of degree at most n to f in the equidistant nodes

$$x_k^{(n)} := -1 + 2k/n, \qquad k = 0, 1, ..., n. \tag{1.1}$$

As is well known, the assumption that f is continuous on $[-1, 1]$ does not guarantee that $L_n(f; \cdot)$ converges to f everywhere on $[-1, 1]$ as $n \to \infty$. Indeed, Bernstein (cf. [N, p. 30]) proved that for $f(x) = |x|$, the sequence $L_n(f; x)$ converges only at the three points $x = -1, 0$, and 1. (For further results concerning this example, see [BMS, LM].)

Moreover, even if f is assumed to be analytic on an open set of the complex plane $\mathbf{C}$ that contains $[-1, 1]$, Runge (cf. [IK, Chap. 6, Sect. 3.6]) showed that the convergence of $L_n(f; \cdot)$ may hold only on a proper

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subinterval of $[-1, 1]$ and at the endpoints ± 1. To describe the region of convergence in this situation, we introduce the potential corresponding to the uniform distribution on $[-1, 1]$:

$$U(z) := \frac{1}{2} \int_{-1}^{1} \log|z-t| \, dt. \tag{1.2}$$

Let Γ_s denote generically the level curves

$$\Gamma_s = \{z \in \mathbf{C} : U(z) = U(s)\}, \qquad s > 0, \tag{1.3}$$

and G_s denote the interior of Γ_s:

$$G_s := \{z \in \mathbf{C} : U(z) < U(s)\}. \tag{1.4}$$

As is easily seen (cf. Lemma 3.1), the level curves Γ_s degenerate to the single point $z = 0$ as $s \to 0^+$; moreover, the sets G_s are Jordan regions that expand as s increases (see Fig. 1.1).

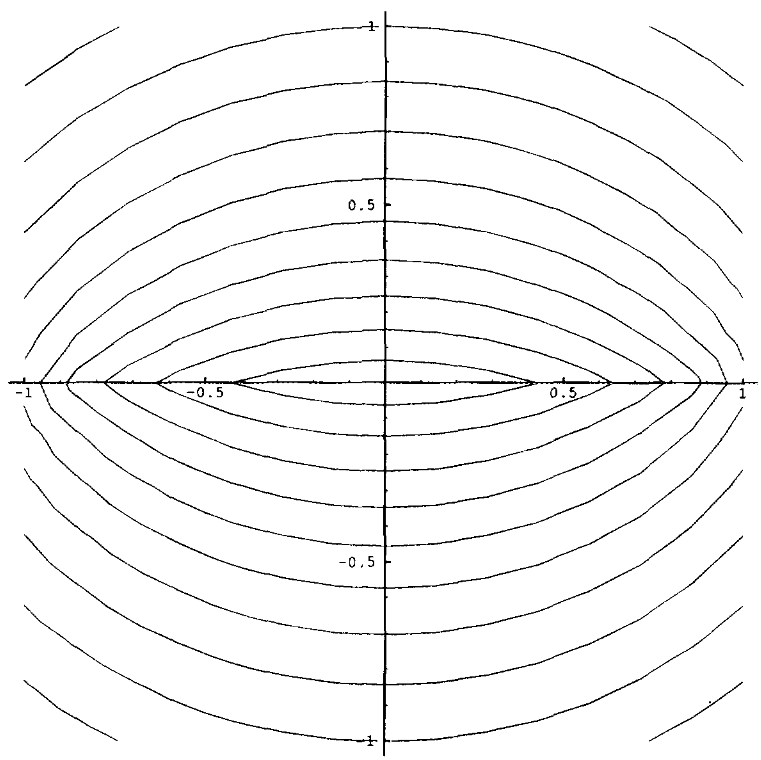

FIG. 1.1. Level curves Γ_s.

The following theorem is an application of classical results on interpolating polynomials.

THEOREM 1.1 [K, WAR]. *Assume that f is analytic on $[-1, 1]$. If f is also analytic in the region G_s of* (1.4) *for some $s>0$, then the Lagrange interpolating polynomials $L_n(f;\cdot)$ to f in equidistant points converge locally uniformly to f in G_s.*

Note that if $s<1$, then G_s does not contain the whole interval $[-1, 1]$ and for real x, Theorem 1.1 only implies convergence on the proper subinterval $(-U(s), U(s))$ of $[-1, 1]$. This behavior is certainly consistent with Runge's example.

What seems to be extraneous in Theorem 1.1 is the assumption that f is analytic everywhere on $[-1, 1]$. Yet this hypothesis appears to be crucial in the proof of Theorem 1.1, where the global analyticity of f is needed in order to apply the Hermite integral representation formula for the interpolating polynomials.

The aim of the present paper is to show that, indeed, the conclusion of Theorem 1.1 remains valid if f is only assumed to be bounded on $[-1, 1]$ and analytic in G_s. Consequently, if f is analytic at $x=0$ and bounded on $[-1, 1]$ (e.g., $f(x)=|x-s|$, $s>0$), then the Lagrange interpolants $L_n(f;\cdot)$ converge to f in a neighborhood of the origin. We prove this fact by employing a local version of the Hermite representation formula—a procedure that can likely be applied in studying other interpolation schemes.

Our main results, which are stated in Section 2, also include a generalization to interpolation in equidistant points by rational functions with fixed denominator degree. Thereby we obtain a local version of a de Montessus de Ballore type theorem in the same spirit as Warner's generalization [War] of a theorem due to the second author [S]. The proofs of the main results are given in Section 3. In Section 4 we mention some open problems and conjectures related to our investigations.

2. MAIN RESULTS

With the notation of the Introduction we now state our main results. The proofs are given in Section 3.

THEOREM 2.1. *Assume that the function f is bounded on $[-1, 1]$ and analytic at $x=0$. Let σ be the largest positive number s such that f has an analytic continuation to the region G_s of* (1.4). *Then the Lagrange interpolants $L_n(f;\cdot)$ in equidistant points converge locally uniformly to f in G_σ.*

More precisely, for each $s \in (0, \sigma)$,

$$\limsup_{n \to \infty} \|L_n(f; \cdot) - f(\cdot)\|_{\Gamma_s}^{1/n} \leqslant e^{U(s) - U(\sigma)} < 1, \tag{2.1}$$

where $\|\cdot\|_{\Gamma_s}$ *denotes the uniform norm on the level curve* Γ_s *of* (1.3).

Remark 1. It is straightforward to verify that for $s > 0$, the function $U(s)$ of (1.2) is increasing and continuous and satisfies

$$e^{U(s)} = \frac{(1+s)^{(1+s)/2}\,|1-s|^{(1-s)/2}}{e}, \qquad s > 0. \tag{2.2}$$

Thus the geometric rate of convergence in (2.1) can be expressed as

$$e^{U(s) - U(\sigma)} = \frac{(1+s)^{(1+s)/2}\,|1-s|^{(1-s)/2}}{(1+\sigma)^{(1+\sigma)/2}\,|1-\sigma|^{(1-\sigma)/2}}. \tag{2.3}$$

Remark 2. It is easy to show (cf. [WAL]) that the equality holds in (2.1) if $\sigma > 1$ and $\sigma > s \geqslant 1$. However, we do not know whether equality also holds if $s < 1$ or $\sigma < 1$.

Remark 3. On the whole level curve Γ_σ the polynomials $L_n(f; \cdot)$ cannot converge at a geometric rate because otherwise f could be extended as an analytic function to a larger region G_τ, $\tau > \sigma$ (cf. [WAL]).

An interesting consequence of the proof of Theorem 2.1 is the following.

COROLLARY 2.2. *Let* a_n *be the leading coefficient* (*i.e., the coefficient of the* x^n *term*) *of the polynomial* $L_n(f; \cdot)$. *Then under the hypotheses of Theorem* 2.1, *there holds*

$$\limsup_{n \to \infty} |a_n|^{1/n} \leqslant e^{-U(\sigma)}. \tag{2.4}$$

Remark 4. Corollary 2.2 provides estimates for the divided difference of f in the points $x_k^{(n)}$ of (1.1) since $a_n = f[x_0^{(n)}, x_1^{(n)}, ..., x_n^{(n)}]$.

Theorem 2.1 has an extension to interpolation by rational functions having fixed denominator degree as we now describe. We call $r_{n,\nu}(z)$ a *rational function of type* (n, ν) if it is of the form

$$r_{n,\nu}(z) = p_n(z)/q_\nu(z), \qquad q_\nu(z) \not\equiv 0,$$

where $p_n \in \mathscr{P}_n$ (the set of polynomials of degree at most n) and $q_\nu(z) \in \mathscr{P}_\nu$. The classical result of de Montessus de Ballore [M] concerns the convergence of rational functions that interpolate maximally in the origin (i.e., Padé approximants) a function f that is analytic at $z = 0$ and meromorphic with precisely ν poles (counting multiplicity) in some disk $|z| < \tau$. For the sequence of Padé approximants of type (n, ν), $n = 0, 1, 2, \ldots$, de Montessus de Ballore proved that convergence holds locally uniformly in the disk $|z| < \tau$ with the ν poles of f deleted.

The second author [S] showed that the above result has a generalization to rational functions that interpolate in a triangular scheme of points whose normalized counting measures converge in the weak-star sense to the equilibrium distribution of a compact set E of the complex plane. Warner [WAR] then showed that this result holds even if the limit distribution is not an equilibrium distribution. In particular, Warner obtained a de Montessus de Ballore type theorem that applies to interpolation by rational functions in equidistant nodes, *provided that the interpolated function f is analytic on* $[-1, 1]$. But the method used in the proof of Theorem 2.1 can be adapted to prove the following local version of that result.

THEOREM 2.3. *Let f be bounded on $[-1, 1]$ and meromorphic with precisely ν poles $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ in G_σ for some $\sigma > 0$. Assume that the poles α_j are different from the nodes $\{x_k^{(m)}\}_{m=1}^{\infty}$ of* (1.1). *Then for each n sufficiently large, there exists a (unique) rational function $r_{n\nu}(z) = p_{n\nu}(z)/q_{n\nu}(z)$ of type (n, ν) that interpolates f in the $n+\nu+1$ points $\{x_k^{(n+\nu)}\}_{k=0}^{n+\nu}$. As $n \to \infty$, the sequence $r_{n\nu}(z)$ converges to $f(z)$ locally uniformly in the region $D_\sigma := G_\sigma \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_\nu\}$; more precisely, if $E \subset D_\sigma$ is compact, then*

$$\limsup_{n \to \infty} \|f - r_{n\nu}\|_E^{1/n} \leqslant \max_{z \in E} e^{U(z) - U(\sigma)} < 1. \tag{2.5}$$

Furthermore, if the denominator polynomials $q_{n\nu}$ are normalized to be monic, and $q(z) := \prod_{i=1}^{\nu} (z - \alpha_i)$, then

$$\limsup_{n \to \infty} \|q_{n\nu} - q\|_K^{1/n} \leqslant \max_{1 \leqslant i \leqslant \nu} e^{U(\alpha_i) - U(\sigma)} < 1, \tag{2.6}$$

for every compact set K of the plane.

3. PROOFS

For proofs of the results in Section 2 we make use of the following simple lemmas.

LEMMA 3.1. *Let $U(z)$ be defined as in* (1.2). *Then*

(i) *$U(z)$ is even and continuous in* $\mathbf{C}$;

(ii) *for $s > 0$, $U(s)$ is increasing*;

(iii) *for each* $s>0$, *the level curve* Γ_s *is a single Jordan curve and for* $0<s_1<s_2$, $\Gamma_{s_1}\subset G_{s_2}$.

Proof. Assertions (i) and (ii) follow from elementary calculation (cf. (2.2)); (iii) is a consequence of (i), (ii), and the fact that $U(z)$ is a harmonic function in $\mathbf{C}\setminus[-1,1]$. ∎

LEMMA 3.2. *For* $x\in(-1,1)$, *define*

$$k(x):=\max\{k\in\{0,1,...,n\}\mid -1+2k/n\leqslant x\}. \tag{3.1}$$

Then

$$\lim_{n\to\infty}\binom{n}{k(x)}^{1/n}=\frac{2}{(1+x)^{(1+x)/2}(1-x)^{(1-x)/2}}.$$

Proof. Note that $k(x)=[(x+1)n/2]$, where $[\cdot]$ denotes the greatest integer function. The proof then follows by using Stirling's formula. ∎

An important role in our proofs is played by the fundamental polynomials

$$w_n(z):=\prod_{k=0}^{n}(z-x_k^{(n)}), \tag{3.2}$$

where the $x_k^{(n)}$'s are defined in (1.1).

LEMMA 3.3. (i) *The polynomials* $w_n(z)$ *satisfy*

$$\lim_{n\to\infty}|w_n(z)|^{1/n}=e^{U(z)} \tag{3.3}$$

locally uniformly in $\mathbf{C}\setminus[-1,1]$.

(ii) *For every fixed* $x\in\mathbf{R}$, $|w_n(x+iy)|$ *is an increasing function for* $y\geqslant 0$.

Proof. Assertion (i) is straightforward to establish on considering $(1/n)\log|w_n(z)|$ for $z\notin[-1,1]$. Assertion (ii) follows from the representation

$$|w_n(x+iy)|^2=\prod_{k=0}^{n}((x-x_k^{(n)})^2+y^2). \quad ∎$$

The following lemma is contained in Corollary 4 and Lemma 5 of [LM] and is also implied by an earlier result of Davis and Rabinowitz [DR] on approximate singular integration. Here, for the convenience of the reader, we give a direct proof.

LEMMA 3.4. *The limit in* (3.3) *holds for almost all* $x \in [-1, 1]$.

Proof. Let $x \in [-1, 1] \setminus \{x_k^{(n)}\}_{k=0,\, n=1}^{n;\, \infty}$ and write

$$\log |w_n(x)|^{1/n} = \frac{1}{n} \sum_{k=0}^{n} \log |x - x_k^{(n)}|.$$

Then by the monotonicity of $\log|x-t|$ in t on the intervals $[-1, x)$ and $(x, 1]$, we have

$$\frac{1}{n} \sum_{k=0}^{k(x)-1} \log |x - x_k^{(n)}| \geqslant \frac{1}{2} \int_{-1}^{x_{k(x)}^{(n)}} \log |x-t|\, dt$$

$$\geqslant \frac{1}{n} \sum_{k=1}^{k(x)-1} \log |x - x_k^{(n)}| + \frac{1}{2} \int_{x_{k(x)-1}^{(n)}}^{x_{k(x)}^{(n)}} \log |x-t|\, dt,$$

where $k(x)$ is defined in (3.1). Consequently,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{k(x)-1} \log |x - x_k^{(n)}| = \frac{1}{2} \int_{-1}^{x} \log |x-t|\, dt.$$

Similarly, we can verify that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=k(x)+2}^{n} \log |x - x_k^{(n)}| = \frac{1}{2} \int_{x}^{1} \log |x-t|\, dt.$$

Thus, to complete the proof, we need only show that

$$\lim_{n \to \infty} \frac{1}{n} \log |(x - x_{k(x)}^{(n)})(x - x_{k(x)+1}^{(n)})| = 0$$

for almost all $x \in [-1, 1]$. Note that this is equivalent to

$$\lim_{n \to \infty} \frac{1}{n} \log(\min_{0 \leqslant k \leqslant n} |x - x_k^{(n)}|) = 0$$

for almost all $x \in [-1, 1]$. Consequently, on setting

$$S := \left\{ x \in [-1, 1] \mid \liminf_{n \to \infty} \frac{1}{n} \log(\min_{0 \leqslant k \leqslant n} |x - x_k^{(n)}|) < 0 \right\},$$

our proof will be complete if we show that S is of Lebesgue measure zero. But this follows immediately from the representation

$$S = \bigcup_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \bigcup_{k=0}^{n} \left(x_k^{(n)} - \left(1 - \frac{1}{m}\right)^n, x_k^{(n)} + \left(1 - \frac{1}{m}\right)^n \right). \quad \blacksquare$$

We are now ready to prove Theorem 2.1. It suffices to establish the theorem only for the case when $\sigma \leqslant 1$, since the case when $\sigma > 1$ can be obtained from Warner's result [WAR]. (Note that when $\sigma > 1$, the function f is analytic on $[-1, 1]$.)

Proof of Theorem 2.1. For $s_0 \in (0, \sigma)$, select s_1, x, and s_2 with x satisfying (3.3) (by Lemma 3.4 this is possible) and

$$s_0 < s_1 < x < s_2 < \sigma \leqslant 1.$$

Note that $x \neq x_k^{(n)}$, $k = 0, ..., n$; $n = 1, 2, ...$ Let $\delta_1 > 0$ and $\delta_2 > 0$ satisfy $s_0 + i\delta_1 \in \Gamma_{s_1}$ and $x + i\delta_2 \in \Gamma_{s_2}$, and define

$$J_1^{\pm} := \{z \in \mathbf{C} \mid z = \pm s_0 + it, |t| \leqslant \delta_1\},$$

$$J_2^{\pm} := \{z \in \mathbf{C} \mid z = \pm x + it, |t| \leqslant \delta_2\}$$

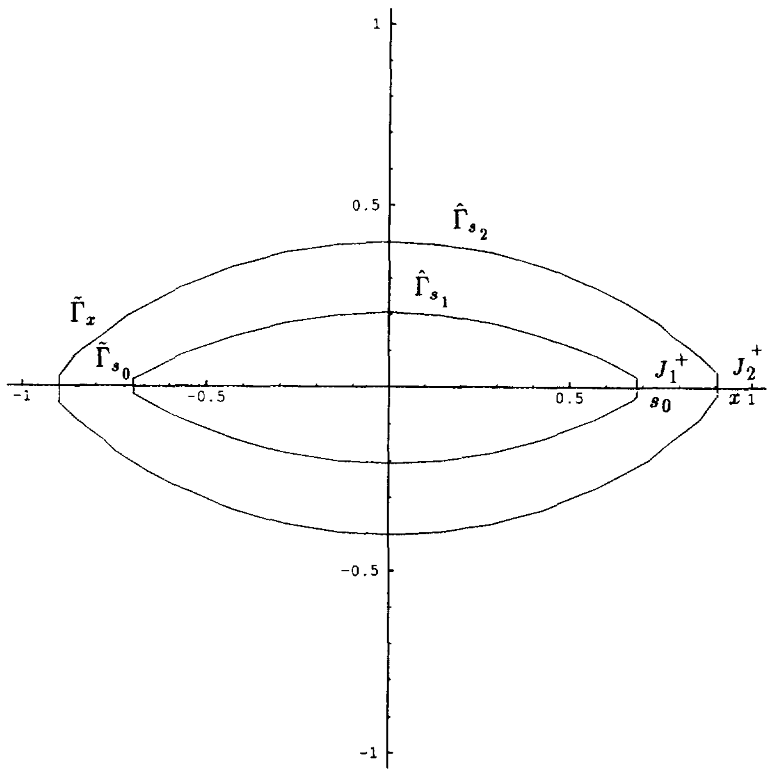

FIG. 3.1. Modified level curves $\tilde{\Gamma}_{s_0}$ and $\tilde{\Gamma}_x$.

and

$$\hat{\Gamma}_{s_1} := \{z \in \Gamma_{s_1} \mid |\operatorname{Re} z| \leqslant s_0\},$$
$$\hat{\Gamma}_{s_2} := \{z \in \Gamma_{s_2} \mid |\operatorname{Re} z| \leqslant x\}.$$

Furthermore, define the modified level curves $\tilde{\Gamma}_{s_0}$ and $\tilde{\Gamma}_x$ as follows (see Fig. 3.1):

$$\tilde{\Gamma}_{s_0} := J_1^+ \cup J_1^- \cup \hat{\Gamma}_{s_1} \quad \text{and} \quad \tilde{\Gamma}_x := J_2^+ \cup J_2^- \cup \hat{\Gamma}_{s_2}.$$

Then Lemma 3.1 (iii) implies that

$$\tilde{\Gamma}_{s_0} \subset \operatorname{Int}(\tilde{\Gamma}_x).$$

For $z \in \tilde{\Gamma}_{s_0}$, write

$$\frac{L_n(f; z) - f(z)}{w_n(z)} = \left(\sum_{|k-(n/2)| \leqslant k(x)-(n/2)} + \sum_{|k-(n/2)| > k(x)-(n/2)} \right) \times \frac{f(x_k^{(n)}) - f(z)}{w_n'(x_k^{(n)})(z - x_k^{(n)})} =: I_1 + I_2.$$

We now estimate $|I_1|$ and $|I_2|$. By the residue theorem, we have

$$I_1 = \sum_{|k-(n/2)| \leqslant k(x)-(n/2)} \frac{f(x_k^{(n)}) - f(z)}{w_n'(x_k^{(n)})(z - x_k^{(n)})} = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_x} \frac{f(\zeta) - f(z)}{w_n(\zeta)(z - \zeta)}\, d\zeta.$$

Set

$$B := \sup\{|f(t)| \mid t \in [-1, 1] \cup \tilde{\Gamma}_x\}.$$

Then it is easy to see that, for $\zeta \in \tilde{\Gamma}_x$, $z \in \tilde{\Gamma}_{s_0}$, we have

$$\left| \frac{f(\zeta) - f(z)}{z - \zeta} \right| \leqslant \frac{2B}{d_{s_0, x}},$$

where here and below $d_{s_0, x} := \inf\{\operatorname{dist}(\alpha, \tilde{\Gamma}_x) : \alpha \in \tilde{\Gamma}_{s_0}\}$. Now by Lemma 3.3(i) we have

$$\lim_{n \to \infty} \min_{\zeta \in \hat{\Gamma}_{s_2}} |w_n(\zeta)|^{1/n} = \exp(U(s_2)).$$

Also, by Lemma 3.3 (ii),

$$|w_n(\zeta)| \geqslant |w_n(x)|$$

for $\zeta \in J_2^+ \cup J_2^-$. Hence we have

$$\limsup_{n \to \infty} |I_1|^{1/n} \leqslant \limsup_{n \to \infty} \max \left\{ \frac{1}{|w_n(x)|^{1/n}}, \frac{1}{\min_{\zeta \in \tilde{\Gamma}_{s_2}} |w_n(\zeta)|^{1/n}} \right\}$$
$$\leqslant \max\{\exp(-U(x)), \exp(-U(s_2))\}.$$

Thus, since $x < s_2$, we get

$$\limsup_{n \to \infty} |I_1|^{1/n} \leqslant \exp(-U(x)) \tag{3.4}$$

uniformly for $z \in \tilde{\Gamma}_{s_0}$.

Now we estimate the sum I_2. For $z \in \tilde{\Gamma}_{s_0}$,

$$|I_2| = \left| \sum_{|k-n/2| > k(x) - n/2} \frac{f(x_k^{(n)}) - f(z)}{w_n'(x_k^{(n)})(x_k^{(n)} - z)} \right|$$
$$\leqslant \frac{2B}{d_{s_0, x}} \sum_{|k-n/2| > k(x) - n/2} \frac{1}{|w_n'(x_k^{(n)})|}.$$

But by straightforward computation,

$$w_n'(x_k^{(n)}) = \left(\frac{2}{n}\right)^n (-1)^{n-k} k!(n-k)!, \qquad k = 0, 1, ..., n,$$

and so

$$|I_2| \leqslant \frac{2B}{d_{s_0, x}} \sum_{|k-n/2| > k(x) - n/2} \frac{1}{(2/n)^n k!(n-k)!}$$
$$= \frac{2B(n/2)^n}{n!\, d_{s_0, x}} \sum_{|k-n/2| > k(x) - n/2} \binom{n}{k}$$
$$\leqslant \frac{2B(n/2)^n}{n!\, d_{s_0, x}} 2(n - k(x)) \binom{n}{k(x)}.$$

Now using Stirling's formula, Lemma 3.2 and (2.2), we obtain

$$\limsup_{n \to \infty} |I_2|^{1/n} \leqslant \frac{e}{(1+x)^{(1+x)/2} (1-x)^{(1-x)/2}} = \exp(-U(x)). \tag{3.5}$$

Hence, from (3.4) and (3.5), we have

$$\limsup_{n \to \infty} \left| \frac{L_n(f; z) - f(z)}{w_n(z)} \right|^{1/n} \leqslant \exp(-U(x)), \tag{3.6}$$

uniformly for $z \in \tilde{\Gamma}_{s_0}$.

Since Lemma 3.3(i) implies that

$$\lim_{n\to\infty} |w_n(z)|^{1/n} = \exp(U(s_1)),$$

uniformly for $z \in \hat{\Gamma}_{s_1}$, and Lemma 3.3(ii) implies

$$|w_n(z)| \leqslant |w_n(s_0 + i\delta_1)|$$

for all $z \in J_1^+ \cup J_1^-$, we have that

$$\limsup_{n\to\infty} |w_n(z)|^{1/n} \leqslant \exp(U(s_1))$$

uniformly for $z \in \tilde{\Gamma}_{s_0}$. With (3.6), it then follows that

$$\limsup_{n\to\infty} |L_n(f; z) - f(z)|^{1/n} \leqslant \exp(U(s_1) - U(x))$$

uniformly for $z \in \tilde{\Gamma}_{s_0}$ and so, by the maximum modulus principle, we get

$$\limsup_{n\to\infty} \|L_n(f; \cdot) - f(\cdot)\|_{\Gamma_{s_0}}^{1/n} \leqslant \exp(U(s_1) - U(x)).$$

Finally, letting $s_1 \to s_0^+$ and $x \to \sigma^-$ gives

$$\limsup_{n\to\infty} \|L_n(f; \cdot) - f(\cdot)\|_{\Gamma_{s_0}}^{1/n} \leqslant \exp(U(s_0) - U(\sigma)) < 1.$$

This completes the proof of Theorem 2.1. ∎

Proof of Corollary 2.2. Recall that a_n is the nth divided difference of f associated with $\{x_k^{(n)}\}_{k=0}^n$. So we can write

$$a_n = \sum_{k=0}^n \frac{f(x_k^{(n)})}{w_n'(x_k^{(n)})}.$$

Now the proof of Corollary 2.2 follows from the same argument as in the proof of Theorem 2.1. ∎

Proof of Theorem 2.3. It is known that there are polynomials $P_{n\nu} \in \mathscr{P}_n$ and $Q_{n\nu}(\not\equiv 0) \in \mathscr{P}_\nu$ such that $Q_{n\nu}f - P_{n\nu}$ vanishes in the zeros of $w_{n+\nu}(z)$. Hence $qP_{n\nu} \in \mathscr{P}_{n+\nu}$ interpolates $qQ_{n\nu}f$ in the points $\{x_k^{(n+\nu)}\}_{k=0}^{n+\nu}$; that is, $qP_{n+\nu} = L_{n+\nu}(qQ_{n\nu}f; \cdot)$. On multiplying by a suitable constant, we can assume that the maximum modulus of the coefficients of $Q_{n\nu}$ is 1 and that its leading coefficient is positive. Then $qQ_{n\nu}f$ is analytic in G_σ and bounded (uniformly in n) on $[-1, 1] \cup \Gamma_s$ for each fixed $s < \sigma$. Now using the same argument as in the proof of Theorem 2.1, we get

$$\limsup_{n\to\infty} \|qQ_{n\nu}f - qP_{n\nu}\|_{\Gamma_{s_0}}^{1/(n+\nu)} \leqslant e^{U(s_0) - U(\sigma)} < 1, \tag{3.7}$$

for $s_0 \in (0, \sigma)$. Then the inequality

$$\limsup_{n \to \infty} |Q_{n\nu}(\alpha_i)|^{1/n} \leqslant e^{U(\alpha_i) - U(\sigma)} < 1$$

follows by choosing s_0 so that $U(s_0) = U(\alpha_i)$ and using the facts that $q(\alpha_i) = 0$ and $(qf)(\alpha_i) \neq 0$. In case f has a pole order m at α_i a similar estimate holds for $Q_{n\nu}^{(j)}(\alpha_i)$, $j = 0, 1, ..., m-1$. Then by a standard argument (see, for example, [GS]) one can show that

$$c_n := \text{the leading coefficient of } Q_{n\nu} \to 1 \qquad (n \to \infty)$$

and

$$\limsup_{n \to \infty} \|Q_{n\nu} - c_n q\|_K^{1/n} \leqslant \max_{1 \leqslant i \leqslant \nu} e^{U(\alpha_i) - U(\sigma)} < 1,$$

for any compact set $K \subset \mathbf{C}$. Letting $q_{n\nu}(z) = Q_{n\nu}(z)/c_n$ and $p_{n\nu}(z) = P_{n\nu}(z)/c_n$, we obtain (2.6). Finally, (2.5) is implied by (2.6) and (3.7). ∎

4. Open Problems and Conjectures

In this section we list some problems and conjectures suggested by the results of this paper.

Problem 4.1. With the hypotheses of Theorem 2.1, what can be said about the divergence of the sequence $L_n(f; z)$ for z outside Γ_σ?

Based on some example cases and the situation for $\sigma > 1$, it seems likely that divergence holds at all non-real points outside Γ_σ.

Problem 4.2. Under the assumptions of Theorem 2.1, study the limiting distribution of the zeros $L_n(f; z)$ as $n \to \infty$.

For the function $f(x) = |x - \sigma|$ with $\sigma \in (-1, 1)$, the following result is obtained in [LS].

THEOREM 4.3. [LS]. *Let $f(x) = |x - \sigma|$ for fixed $\sigma \in (-1, 1)$. Then the normalized counting measures of the zeros of $L_n(f; \cdot)$ converge, in the weak-star topology, to the balayage of the uniform distribution $\chi_{[-1,1]}\, dt/2$ to the boundary of $[-1, 1] \cup G_\sigma$ for almost every σ.*

Concerning Problem 4.2, in view of Theorem 4.3 it is quite tempting to make the following conjecture.

Conjecture 4.4. In addition to the hypotheses of Theorem 2.1, assume that f does not vanish identically on G_σ. Then there is a subsequence of the normalized counting measures of $L_n(f; \cdot)$ that converges, in the weak-star topology, to the balayage of the uniform distribution $\chi_{[-1,1]}\,dt/2$ to the boundary of $[-1, 1] \cup G_\sigma$.

It seems that in order to solve Conjecture 4.4, it would be very helpful to first prove the following.

Conjecture 4.5. Under the assumption of Theorem 2.1, the equality holds in (2.4).

Since a_n is the nth divided difference of f associated with equidistant nodes, a proof of Conjecture 4.5 will yield some interesting results on the totality of the divided difference functionals associated with equidistant nodes (cf. [IRS]).

Finally, we pose

Problem 4.6. Characterize the sets of nodes of interpolation that have local convergence properties similar to those established in Theorem 2.1.

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